The symmetries of the Fokker - Planck equation in two dimensions

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ABSTRACT

We calculate all point symmetries of the Fokker - Planck equation in two-dimensional Euclidean space. General expression of symmetry group action on arbitrary solution of Fokker - Planck equation is presented.

1. The symmetries of the Fokker - Planck equation in two dimensions

The object of our considerations is a special case of Fokker - Planck equation, which describes evolution of 2D continuum of non-interacting particles imbedded in a dense medium without outer forces. The interaction between particles and medium causes combined diffusion in physical space and velocities space. The only force, which acts on particles, is damping force proportional to velocity.

The 3D variant of this equation was investigated in our work [1]. In this work fundamental solution of 3D equation was obtained by means of Fourier transform.

The 1D variant of this equation was investigated in our work [2]. All point symmetries of the Fokker - Planck equation in one-dimensional Euclidean space were calculated.

In present work we continue this investigation for more complex 2D equation.

The Fokker - Planck equation in two dimensions is

$$\frac{\partial n}{\partial t} + u\,\frac{\partial n}{\partial x} + v\,\frac{\partial n}{\partial y} - au\,\frac{\partial n}{\partial u} - av\,\frac{\partial n}{\partial v} - 2an - k\left(\frac{\partial^2 n}{\partial u^2} + \frac{\partial^2 n}{\partial v^2}\right) = 0; \qquad (1)$$

where

$n = n(t, x, y, u, v)$ - density;

t - time variable;

x, y - space coordinates;

u, v - velocity;

a - coefficient of damping;

k - coefficient of diffusion.

The list of symmetries of the Fokker - Planck equation in one dimension follows. The calculations of symmetries are rather awkward. They are carried out to APPENDIX 1.

Instead of classic "$\xi - \phi$" notation we use another ("δ") notation. This notation was presented in our work [2].

Addition of arbitrary solution

$$\mathbf{v}_1 = A\,\frac{\partial}{\partial n}\,; \qquad (2)$$

where A is arbitrary solution of the (1) equation.

Scaling of density

$$\mathbf{v}_2 = n\frac{\partial}{\partial n}; \tag{3}$$

The reason of symmetries (2-3) existence is linearity of PDE (1).

Time shift

$$\mathbf{v}_3 = \frac{\partial}{\partial t}; \tag{4}$$

Space translations

$$\mathbf{v}_4 = \frac{\partial}{\partial x};\ \ \mathbf{v}_5 = \frac{\partial}{\partial y}. \tag{5}$$

Space rotation

$$\mathbf{v}_6 = v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v} + y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}. \tag{6}$$

Transformations (5) and (6) build two-dimensional Euclidean movements group.

Extended Galilean transformations, which besides time and space coordinates affect the density

$$\mathbf{v}_7 = \frac{\partial}{\partial u} + t\frac{\partial}{\partial x} - \frac{an}{2k}(ax+u)\frac{\partial}{\partial n};\ \ \mathbf{v}_8 = \frac{\partial}{\partial v} + t\frac{\partial}{\partial x} - \frac{an}{2k}(ax+v)\frac{\partial}{\partial n}. \tag{7}$$

Negative exponent transformations - they affect time and space coordinates, contain time-dependent common multiplier (negative exponent). They do not affect density.

$$\mathbf{v}_9 = e^{-at}\left(-a\frac{\partial}{\partial u} + \frac{\partial}{\partial x}\right);\ \ \mathbf{v}_{10} = e^{-at}\left(-a\frac{\partial}{\partial v} + \frac{\partial}{\partial y}\right). \tag{8}$$

Positive exponent transformations - they affect time, space and density, contain time-dependent common multiplier (positive exponent).

$$\mathbf{v}_{11} = e^{at}\left(a\frac{\partial}{\partial u} + \frac{\partial}{\partial x} - \frac{a^2}{k}nu\frac{\partial}{\partial n}\right);\ \ \mathbf{v}_{12} = e^{at}\left(a\frac{\partial}{\partial v} + \frac{\partial}{\partial y} - \frac{a^2}{k}nv\frac{\partial}{\partial n}\right). \tag{9}$$

One-parameter groups, generated by vector fields $\mathbf{v}_1 - \mathbf{v}_{12}$, are enumerated in the following list. The list contains images of the point (n, t, x, y, u, v) by transformation $exp(\varepsilon\mathbf{v}_i)$

$G_{1:}$ $(n+\varepsilon A, t, x, y, u, v)$;

$G_{2:}$ $(e^{\varepsilon}n, t, x, y, u, v)$;

$G_{3:}$ $(n, t+\varepsilon, x, y, u, v)$;

$G_{4:}$ $(n, t, x+\varepsilon, y, u, v)$;

$G_{5:}$ $(n, t, x, y+\varepsilon, u, v)$;

$G_{6:}$ $(n, t, \cos(\varepsilon)x + \sin(\varepsilon)y, -\sin(\varepsilon)x + \cos(\varepsilon)y, \cos(\varepsilon)u + \sin(\varepsilon)v, -\sin(\varepsilon)u + \cos(\varepsilon)v)$; (10)

$G_{7:}$ $(\exp\left[-\frac{a}{2k}\left(\varepsilon(ax+u) + \frac{1}{2}\varepsilon^2(at+1)\right)\right]n, t, x+\varepsilon t, y, u+\varepsilon, v)$;

$G_{8:}$ $\quad (\exp\left[-\frac{a}{2k}\left(\varepsilon(ay+v)+\frac{1}{2}\varepsilon^2(at+1)\right)\right]n, t, x, y+\varepsilon t, u, v+\varepsilon);$

$G_{9:}$ $\quad (n, t, x+\varepsilon e^{-at}, y, u-\varepsilon a\ e^{-at}, v);$

$G_{10:}$ $\quad (n, t, x, y+\varepsilon e^{-at}, u, v-\varepsilon a\ e^{-at});$

$G_{11:}$ $\quad (n\ \exp\left[-\frac{a^2}{k}e^{at}\left(\varepsilon u+\frac{1}{2}\varepsilon^2 a\ e^{at}\right)\right], t, x+\varepsilon e^{at}, y, u+\varepsilon a\ e^{at}, v).$

$G_{12:}$ $\quad (n\ \exp\left[-\frac{a^2}{k}e^{at}\left(\varepsilon v+\frac{1}{2}\varepsilon^2 a\ e^{at}\right)\right], t, x, y+\varepsilon e^{at}, u, v+\varepsilon a\ e^{at}).$

For relatively nontrivial integration of G_7 , G_8 , G_{11} and G_{12} we refer to [2].

The fact, that G_i are symmetries of PDE (1) means, that if $f(t, x, y, u, v)$ is arbitrary solution of (1), the functions

$u^{(1)}$: $\quad f(t, x, y, u, v)+\varepsilon A(t, x, y, u, v);$

$u^{(2)}$: $\quad e^{\varepsilon} f(t, x, y, u, v);$

$u^{(3)}$: $\quad f(t-\varepsilon, x, y, u, v);$

$u^{(4)}$: $\quad f(t, x-\varepsilon, y, u, v);$

$u^{(5)}$: $\quad f(t, x, y-\varepsilon, u, v);$

$u^{(6)}$: $\quad f(t, \cos(\varepsilon)x-\sin(\varepsilon)y, \sin(\varepsilon)x+\cos(\varepsilon)y, \cos(\varepsilon)u-\sin(\varepsilon)v, \sin(\varepsilon)u+\cos(\varepsilon)v);$

$u^{(7)}$: $\quad \exp\left[-\frac{a}{2k}\left(\varepsilon(ax+u)-\frac{1}{2}\varepsilon^2(at+1)\right)\right]f(t, x-\varepsilon t, y, u-\varepsilon, v);$ (11)

$u^{(8)}$: $\quad \exp\left[-\frac{a}{2k}\left(\varepsilon(ay+v)-\frac{1}{2}\varepsilon^2(at+1)\right)\right]f(t, x, y-\varepsilon t, u, v-\varepsilon);$

$u^{(9)}$: $\quad f(t, x-\varepsilon e^{-at}, y, u+\varepsilon a\ e^{-at}, v);$

$u^{(10)}$: $\quad f(t, x, y-\varepsilon e^{-at}, u, v+\varepsilon a\ e^{-at});$

$u^{(11)}$: $\quad \exp\left[-\frac{a^2}{k}e^{at}\left(\varepsilon u-\frac{1}{2}\varepsilon^2 a\ e^{at}\right)\right] f(t, x-\varepsilon e^{at}, y, u-\varepsilon a e^{at}, v).$

$u^{(12)}$: $\quad \exp\left[-\frac{a^2}{k}e^{at}\left(\varepsilon v-\frac{1}{2}\varepsilon^2 a\ e^{at}\right)\right] f(t, x, y-\varepsilon e^{at}, u, v-\varepsilon a e^{at}).$

where ε - arbitrary real number , also are solutions of (1). Here A is another arbitrary solution of (1).

We systematically replaced "old coordinates" by their expressions through "new coordinates". Note, that due to these replacements terms with ε^2 in $u^{(7)}$, $u^{(8)}$, $u^{(11}$ and $u^{(12)}$ change their signs.

We have trivial solution $n = e^{2at}$ at our disposal. If we act on this solution by transformations (10), we obtain 4 new solutions:

$$n = \exp\left[2at - \frac{a}{2k}\left(\varepsilon(ax+u) - \frac{1}{2}\,\varepsilon^2(at+1)\right)\right]; \tag{12}$$

$$n = \exp\left[2at - \frac{a}{2k}\left(\varepsilon(ay+v) - \frac{1}{2}\,\varepsilon^2(at+1)\right)\right]. \tag{13}$$

$$n = \exp\left[2at - \frac{a^2}{k}\,e^{at}\left(\varepsilon u - \frac{1}{2}\,\varepsilon^2 a\,e^{at}\right)\right]; \tag{14}$$

$$n = \exp\left[2at - \frac{a^2}{k}\,e^{at}\left(\varepsilon v - \frac{1}{2}\,\varepsilon^2 a\,e^{at}\right)\right]. \tag{15}$$

General expression is

$$U = e^{\varepsilon_2}\exp\left[-\frac{a}{2k}\left(\varepsilon_7(a\bar{x}+\bar{u}) - \frac{1}{2}\,\varepsilon_7^2(at+1)\right)\right]\exp\left[-\frac{a}{2k}\left(\varepsilon_8(a\bar{y}+\bar{v}) - \frac{1}{2}\,\varepsilon_8^2(at+1)\right)\right]\times \tag{16}$$

$$\times\exp\left[-\frac{a^2}{k}\,e^{at}\left(\varepsilon_{11}(\bar{u}-\varepsilon_7) - \frac{1}{2}\,\varepsilon_{11}^2 a\,e^{at}\right)\right]\exp\left[-\frac{a^2}{k}\,e^{at}\left(\varepsilon_{12}(\bar{v}-\varepsilon_8) - \frac{1}{2}\,\varepsilon_{12}^2 a\,e^{at}\right)\right]\times$$

$$\times f(t-\varepsilon_3, \bar{x}-\varepsilon_4-\varepsilon_7 t-\varepsilon_9 e^{-at}-\varepsilon_{11}e^{at}, \bar{y}-\varepsilon_5-\varepsilon_8 t-\varepsilon_{10}e^{-at}-\varepsilon_{12}e^{at}, \bar{u}-\varepsilon_7+\varepsilon_9 a\,e^{-at}-\varepsilon_{11}a\,e^{at}, \bar{v}-\varepsilon_8+\varepsilon_{10}a\,e^{-at}-\varepsilon_{12}a\,e^{at})+$$

$$+\varepsilon_1 A(t,x,y,u,v);$$

where

$$\bar{x} = \cos(\varepsilon_6)x - \sin(\varepsilon_6)y; \tag{17}$$

$$\bar{y} = \sin(\varepsilon_6)x + \cos(\varepsilon_6)y; \tag{18}$$

$$\bar{u} = \cos(\varepsilon_6)u - \sin(\varepsilon_6)v; \tag{19}$$

$$\bar{v} = \sin(\varepsilon_6)u + \cos(\varepsilon_6)v. \tag{20}$$

DISCUSSION

Looking at the list of all point symmetries of the Fokker - Planck equation in two-dimensional Euclidean space, we see, that there is no simple way to get, for example, fundamental solution of PDE, using these symmetries. We have not at our disposal such an instrument, as scaling of independent variables t, x, y, u, v. The result (12-15) of action of symmetry group on trivial solution is not very interesting from physical point of view.

Indirect way of use of Galilean transformations (7) was demonstrated in [1]. The transformation was used for generalisation of solution, which was obtained in the form of exponent of quadratic form of space coordinates and velocities with time dependent coefficients.

There is need of further investigations of Fokker - Planck equation and its set of symmetries, which may lead to another physically interesting results. We can follow the scheme of [7] : to consider invariant solutions for some one-parameter group, thus reduce the independent variables number. To find for obtained in such a way equation all point symmetries - and so long.

In the work [7] this scheme was represented for equations of elasticity and plasticity.

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APPENDIX 1

The infinitesimal invariance criteria for PDE (1) is

$$\delta(\frac{\partial n}{\partial t}) + \delta u \frac{\partial n}{\partial x} + u\delta(\frac{\partial n}{\partial x}) + \delta v \frac{\partial n}{\partial y} + v\delta(\frac{\partial n}{\partial y}) - \tag{A1-1}$$

$$-a\delta u \frac{\partial n}{\partial u} - au\delta(\frac{\partial n}{\partial u}) - a\delta v \frac{\partial n}{\partial v} - av\delta(\frac{\partial n}{\partial v}) - 2a\delta n - k(\delta(\frac{\partial^2 n}{\partial u^2}) + \delta(\frac{\partial^2 n}{\partial v^2})) = 0.$$

According to [2] (APPENDIX 1, eq. (A1-8) and (A1-18)), we have for variations of derivatives following expression:

$$\delta \frac{\partial n}{\partial x} = \frac{\partial}{\partial x}(\delta n) + \frac{\partial}{\partial n}(\delta n)\frac{\partial n}{\partial x} - \frac{\partial n}{\partial x}(\frac{\partial}{\partial x}(\delta x) + \frac{\partial}{\partial n}(\delta x)\frac{\partial n}{\partial x}) - \tag{A1-2}$$

$$-\frac{\partial n}{\partial y}(\frac{\partial}{\partial x}(\delta y) + \frac{\partial}{\partial n}(\delta y)\frac{\partial n}{\partial x}) - \frac{\partial n}{\partial u}(\frac{\partial}{\partial x}(\delta u) + \frac{\partial}{\partial n}(\delta u)\frac{\partial n}{\partial x}) -$$

$$-\frac{\partial n}{\partial v}(\frac{\partial}{\partial x}(\delta v) + \frac{\partial}{\partial n}(\delta v)\frac{\partial n}{\partial x}) - \frac{\partial n}{\partial t}(\frac{\partial}{\partial x}(\delta t) + \frac{\partial}{\partial n}(\delta t)\frac{\partial n}{\partial x});$$

$$\delta \frac{\partial n}{\partial y} = \frac{\partial}{\partial y}(\delta n) + \frac{\partial}{\partial n}(\delta n)\frac{\partial n}{\partial y} - \frac{\partial n}{\partial x}(\frac{\partial}{\partial y}(\delta x) + \frac{\partial}{\partial n}(\delta x)\frac{\partial n}{\partial y}) - \tag{A1-3}$$

$$-\frac{\partial n}{\partial y}(\frac{\partial}{\partial y}(\delta y) + \frac{\partial}{\partial n}(\delta y)\frac{\partial n}{\partial y}) - \frac{\partial n}{\partial u}(\frac{\partial}{\partial y}(\delta u) + \frac{\partial}{\partial n}(\delta u)\frac{\partial n}{\partial y}) -$$

$$-\frac{\partial n}{\partial v}(\frac{\partial}{\partial y}(\delta v) + \frac{\partial}{\partial n}(\delta v)\frac{\partial n}{\partial y}) - \frac{\partial n}{\partial t}(\frac{\partial}{\partial y}(\delta t) + \frac{\partial}{\partial n}(\delta t)\frac{\partial n}{\partial y});$$

$$\delta \frac{\partial n}{\partial u} = \frac{\partial}{\partial u}(\delta n) + \frac{\partial}{\partial n}(\delta n)\frac{\partial n}{\partial u} - \frac{\partial n}{\partial x}(\frac{\partial}{\partial u}(\delta x) + \frac{\partial}{\partial n}(\delta x)\frac{\partial n}{\partial u}) - \tag{A1-4}$$

$$-\frac{\partial n}{\partial y}(\frac{\partial}{\partial u}(\delta y) + \frac{\partial}{\partial n}(\delta y)\frac{\partial n}{\partial u}) - \frac{\partial n}{\partial u}(\frac{\partial}{\partial u}(\delta u) + \frac{\partial}{\partial n}(\delta u)\frac{\partial n}{\partial u}) -$$

$$-\frac{\partial n}{\partial v}(\frac{\partial}{\partial u}(\delta v) + \frac{\partial}{\partial n}(\delta v)\frac{\partial n}{\partial u}) - \frac{\partial n}{\partial t}(\frac{\partial}{\partial u}(\delta t) + \frac{\partial}{\partial n}(\delta t)\frac{\partial n}{\partial u});$$

$$\delta \frac{\partial n}{\partial v} = \frac{\partial}{\partial v}(\delta n) + \frac{\partial}{\partial n}(\delta n)\frac{\partial n}{\partial v} - \frac{\partial n}{\partial x}(\frac{\partial}{\partial v}(\delta x) + \frac{\partial}{\partial n}(\delta x)\frac{\partial n}{\partial v}) - \tag{A1-5}$$

$$-\frac{\partial n}{\partial y}(\frac{\partial}{\partial v}(\delta y) + \frac{\partial}{\partial n}(\delta y)\frac{\partial n}{\partial v}) - \frac{\partial n}{\partial u}(\frac{\partial}{\partial v}(\delta u) + \frac{\partial}{\partial n}(\delta u)\frac{\partial n}{\partial v}) -$$

$$-\frac{\partial n}{\partial v}(\frac{\partial}{\partial v}(\delta v) + \frac{\partial}{\partial n}(\delta v)\frac{\partial n}{\partial v}) - \frac{\partial n}{\partial t}(\frac{\partial}{\partial v}(\delta t) + \frac{\partial}{\partial n}(\delta t)\frac{\partial n}{\partial v});$$

$$\delta \frac{\partial n}{\partial t} = \frac{\partial}{\partial t}(\delta n) + \frac{\partial}{\partial n}(\delta n)\frac{\partial n}{\partial t} - \frac{\partial n}{\partial x}(\frac{\partial}{\partial t}(\delta x) + \frac{\partial}{\partial n}(\delta x)\frac{\partial n}{\partial t}) - \tag{A1-6}$$

$$-\frac{\partial n}{\partial y}(\frac{\partial}{\partial t}(\delta y) + \frac{\partial}{\partial n}(\delta y)\frac{\partial n}{\partial t}) - \frac{\partial n}{\partial u}(\frac{\partial}{\partial t}(\delta u) + \frac{\partial}{\partial n}(\delta u)\frac{\partial n}{\partial t}) -$$

$$-\frac{\partial n}{\partial v}(\frac{\partial}{\partial t}(\delta v)+\frac{\partial}{\partial n}(\delta v)\frac{\partial n}{\partial t})-\frac{\partial n}{\partial t}(\frac{\partial}{\partial t}(\delta t)+\frac{\partial}{\partial n}(\delta t)\frac{\partial n}{\partial t});$$

$$\delta\frac{\partial^2 n}{\partial u^2}=\frac{\partial}{\partial n}(\delta n)\frac{\partial^2 n}{\partial u^2}-\frac{\partial^2 n}{\partial u\partial x}(\frac{\partial}{\partial u}(\delta x)+\frac{\partial}{\partial n}(\delta x)\frac{\partial n}{\partial u})-\frac{\partial^2 n}{\partial u\partial y}(\frac{\partial}{\partial u}(\delta y)+\frac{\partial}{\partial n}(\delta y)\frac{\partial n}{\partial u})- \quad \text{(A1-7)}$$

$$-\frac{\partial^2 n}{\partial u^2}(\frac{\partial}{\partial u}(\delta u)+\frac{\partial}{\partial n}(\delta u)\frac{\partial n}{\partial u})-\frac{\partial^2 n}{\partial u\partial v}(\frac{\partial}{\partial u}(\delta v)+\frac{\partial}{\partial n}(\delta v)\frac{\partial n}{\partial u})-\frac{\partial^2 n}{\partial t\partial u}(\frac{\partial}{\partial u}(\delta t)+\frac{\partial}{\partial n}(\delta t)\frac{\partial n}{\partial u})-$$

$$-\frac{\partial^2 n}{\partial u^2}\frac{\partial n}{\partial x}\frac{\partial}{\partial n}(\delta x)-\frac{\partial^2 n}{\partial u^2}\frac{\partial n}{\partial y}\frac{\partial}{\partial n}(\delta y)-\frac{\partial^2 n}{\partial u^2}\frac{\partial n}{\partial u}\frac{\partial}{\partial n}(\delta u)-\frac{\partial^2 n}{\partial u^2}\frac{\partial n}{\partial v}\frac{\partial}{\partial n}(\delta v)-\frac{\partial^2 n}{\partial u^2}\frac{\partial n}{\partial t}\frac{\partial}{\partial n}(\delta t)+$$

$$+\frac{\partial^2}{\partial u^2}(\delta n)+\frac{\partial^2}{\partial n\partial u}(\delta n)\frac{\partial n}{\partial u}+\frac{\partial n}{\partial u}(\frac{\partial^2}{\partial n\partial u}(\delta n)+\frac{\partial^2}{\partial n^2}(\delta n)\frac{\partial n}{\partial u})-$$

$$-\frac{\partial n}{\partial x}(\frac{\partial^2}{\partial u^2}(\delta x)+\frac{\partial^2}{\partial n\partial u}(\delta x)\frac{\partial n}{\partial u})-\frac{\partial n}{\partial y}(\frac{\partial^2}{\partial u^2}(\delta y)+\frac{\partial^2}{\partial n\partial u}(\delta y)\frac{\partial n}{\partial u})-$$

$$-\frac{\partial n}{\partial u}(\frac{\partial^2}{\partial u^2}(\delta u)+\frac{\partial^2}{\partial n\partial u}(\delta u)\frac{\partial n}{\partial u})-\frac{\partial n}{\partial v}(\frac{\partial^2}{\partial u^2}(\delta v)+\frac{\partial^2}{\partial n\partial u}(\delta v)\frac{\partial n}{\partial u})-$$

$$-\frac{\partial n}{\partial t}(\frac{\partial^2}{\partial u^2}(\delta t)+\frac{\partial^2}{\partial n\partial u}(\delta t)\frac{\partial n}{\partial u})-\frac{\partial n}{\partial x}\frac{\partial n}{\partial u}(\frac{\partial^2}{\partial n\partial u}(\delta x)+\frac{\partial^2}{\partial n^2}(\delta x)\frac{\partial n}{\partial u})-$$

$$-\frac{\partial n}{\partial y}\frac{\partial n}{\partial u}(\frac{\partial^2}{\partial n\partial u}(\delta y)+\frac{\partial^2}{\partial n^2}(\delta y)\frac{\partial n}{\partial u})-\frac{\partial n}{\partial u}\frac{\partial n}{\partial u}(\frac{\partial^2}{\partial n\partial u}(\delta u)+\frac{\partial^2}{\partial n^2}(\delta u)\frac{\partial n}{\partial u})-$$

$$-\frac{\partial n}{\partial v}\frac{\partial n}{\partial u}(\frac{\partial^2}{\partial n\partial u}(\delta v)+\frac{\partial^2}{\partial n^2}(\delta v)\frac{\partial n}{\partial u})-\frac{\partial n}{\partial t}\frac{\partial n}{\partial u}(\frac{\partial^2}{\partial n\partial u}(\delta t)+\frac{\partial^2}{\partial n^2}(\delta t)\frac{\partial n}{\partial u})-$$

$$-\frac{\partial^2 n}{\partial u\partial x}(\frac{\partial}{\partial u}(\delta x)+\frac{\partial}{\partial n}(\delta x)\frac{\partial n}{\partial u})-\frac{\partial^2 n}{\partial u\partial y}(\frac{\partial}{\partial u}(\delta y)+\frac{\partial}{\partial n}(\delta y)\frac{\partial n}{\partial u})-$$

$$-\frac{\partial^2 n}{\partial u^2}(\frac{\partial}{\partial u}(\delta u)+\frac{\partial}{\partial n}(\delta u)\frac{\partial n}{\partial u})-\frac{\partial^2 n}{\partial u\partial v}(\frac{\partial}{\partial u}(\delta v)+\frac{\partial}{\partial n}(\delta v)\frac{\partial n}{\partial u})-\frac{\partial^2 n}{\partial t\partial u}(\frac{\partial}{\partial u}(\delta t)+\frac{\partial}{\partial n}(\delta t)\frac{\partial n}{\partial u});$$

$$\delta\frac{\partial^2 n}{\partial v^2}=\frac{\partial}{\partial n}(\delta n)\frac{\partial^2 n}{\partial v^2}-\frac{\partial^2 n}{\partial v\partial x}(\frac{\partial}{\partial v}(\delta x)+\frac{\partial}{\partial n}(\delta x)\frac{\partial n}{\partial v})-\frac{\partial^2 n}{\partial v\partial y}(\frac{\partial}{\partial v}(\delta y)+\frac{\partial}{\partial n}(\delta y)\frac{\partial n}{\partial v})- \quad \text{(A1-8)}$$

$$-\frac{\partial^2 n}{\partial u\partial v}(\frac{\partial}{\partial v}(\delta u)+\frac{\partial}{\partial n}(\delta u)\frac{\partial n}{\partial v})-\frac{\partial^2 n}{\partial v^2}(\frac{\partial}{\partial v}(\delta v)+\frac{\partial}{\partial n}(\delta v)\frac{\partial n}{\partial v})-\frac{\partial^2 n}{\partial t\partial v}(\frac{\partial}{\partial v}(\delta t)+\frac{\partial}{\partial n}(\delta t)\frac{\partial n}{\partial v})-$$

$$-\frac{\partial^2 n}{\partial v^2}\frac{\partial n}{\partial x}\frac{\partial}{\partial n}(\delta x)-\frac{\partial^2 n}{\partial v^2}\frac{\partial n}{\partial y}\frac{\partial}{\partial n}(\delta y)-\frac{\partial^2 n}{\partial v^2}\frac{\partial n}{\partial u}\frac{\partial}{\partial n}(\delta u)-\frac{\partial^2 n}{\partial v^2}\frac{\partial n}{\partial v}\frac{\partial}{\partial n}(\delta v)-\frac{\partial^2 n}{\partial v^2}\frac{\partial n}{\partial t}\frac{\partial}{\partial n}(\delta t)+$$

$$+\frac{\partial^2}{\partial v^2}(\delta n)+\frac{\partial^2}{\partial n\partial v}(\delta n)\frac{\partial n}{\partial v}+\frac{\partial n}{\partial v}(\frac{\partial^2}{\partial n\partial v}(\delta n)+\frac{\partial^2}{\partial n^2}(\delta n)\frac{\partial n}{\partial v})-$$

$$-\frac{\partial n}{\partial x}(\frac{\partial^2}{\partial v^2}(\delta x)+\frac{\partial^2}{\partial n\partial v}(\delta x)\frac{\partial n}{\partial v})-\frac{\partial n}{\partial y}(\frac{\partial^2}{\partial v^2}(\delta y)+\frac{\partial^2}{\partial n\partial v}(\delta y)\frac{\partial n}{\partial v})-$$

$$-\frac{\partial n}{\partial u}(\frac{\partial^2}{\partial v^2}(\delta u)+\frac{\partial^2}{\partial n\partial v}(\delta u)\frac{\partial n}{\partial v})-\frac{\partial n}{\partial v}(\frac{\partial^2}{\partial v^2}(\delta v)+\frac{\partial^2}{\partial n\partial v}(\delta v)\frac{\partial n}{\partial v})-$$

$$-\frac{\partial n}{\partial t}\left(\frac{\partial^2}{\partial v^2}(\delta t)+\frac{\partial^2}{\partial n\partial v}(\delta t)\frac{\partial n}{\partial v}\right)-\frac{\partial n}{\partial x}\frac{\partial n}{\partial v}\left(\frac{\partial^2}{\partial n\partial v}(\delta x)+\frac{\partial^2}{\partial n^2}(\delta x)\frac{\partial n}{\partial v}\right)-$$

$$-\frac{\partial n}{\partial y}\frac{\partial n}{\partial v}\left(\frac{\partial^2}{\partial n\partial v}(\delta y)+\frac{\partial^2}{\partial n^2}(\delta y)\frac{\partial n}{\partial v}\right)-\frac{\partial n}{\partial u}\frac{\partial n}{\partial v}\left(\frac{\partial^2}{\partial n\partial v}(\delta u)+\frac{\partial^2}{\partial n^2}(\delta u)\frac{\partial n}{\partial v}\right)-$$

$$-\frac{\partial n}{\partial v}\frac{\partial n}{\partial v}\left(\frac{\partial^2}{\partial n\partial v}(\delta v)+\frac{\partial^2}{\partial n^2}(\delta v)\frac{\partial n}{\partial v}\right)-\frac{\partial n}{\partial t}\frac{\partial n}{\partial v}\left(\frac{\partial^2}{\partial n\partial v}(\delta t)+\frac{\partial^2}{\partial n^2}(\delta t)\frac{\partial n}{\partial v}\right)-$$

$$-\frac{\partial^2 n}{\partial v\partial x}\left(\frac{\partial}{\partial v}(\delta x)+\frac{\partial}{\partial n}(\delta x)\frac{\partial n}{\partial v}\right)-\frac{\partial^2 n}{\partial v\partial y}\left(\frac{\partial}{\partial v}(\delta y)+\frac{\partial}{\partial n}(\delta y)\frac{\partial n}{\partial v}\right)-$$

$$-\frac{\partial^2 n}{\partial u\partial v}\left(\frac{\partial}{\partial v}(\delta u)+\frac{\partial}{\partial n}(\delta u)\frac{\partial n}{\partial v}\right)-\frac{\partial^2 n}{\partial v^2}\left(\frac{\partial}{\partial v}(\delta v)+\frac{\partial}{\partial n}(\delta v)\frac{\partial n}{\partial v}\right)-\frac{\partial^2 n}{\partial t\partial v}\left(\frac{\partial}{\partial v}(\delta t)+\frac{\partial}{\partial n}(\delta t)\frac{\partial n}{\partial v}\right);$$

We eliminate $\frac{\partial n}{\partial t}$ in (A1-1) using original equation

$$\frac{\partial n}{\partial t}=-\left(u\frac{\partial n}{\partial x}+v\frac{\partial n}{\partial y}-au\frac{\partial n}{\partial u}-av\frac{\partial n}{\partial v}-2an-k\left(\frac{\partial^2 n}{\partial u^2}+\frac{\partial^2 n}{\partial v^2}\right)\right). \qquad \text{(A1-9)}$$

Collecting similar terms, we obtain following equations:

$\frac{\partial n}{\partial x}\frac{\partial n}{\partial u}$

$$-2ku\frac{\partial^2}{\partial n\partial u}(\delta t)+2k\frac{\partial^2}{\partial n\partial u}(\delta x)=0; \qquad \text{(A1-10)}$$

$\frac{\partial n}{\partial x}\frac{\partial n}{\partial u^2}$

$$-ku\frac{\partial^2}{\partial n^2}(\delta t)+k\frac{\partial^2}{\partial n^2}(\delta x)=0; \qquad \text{(A1-11)}$$

$\frac{\partial n}{\partial x}\frac{\partial n}{\partial v}$

$$-2ku\frac{\partial^2}{\partial n\partial v}(\delta t)+2k\frac{\partial^2}{\partial n\partial v}(\delta x)=0; \qquad \text{(A1-12)}$$

$\frac{\partial n}{\partial x}\frac{\partial n}{\partial v^2}$

$$-ku\frac{\partial^2}{\partial n^2}(\delta t)+k\frac{\partial^2}{\partial n^2}(\delta x)=0; \qquad \text{(A1-13)}$$

$\frac{\partial n}{\partial x}$

$$-auv\frac{\partial}{\partial v}(\delta t)+2aun\frac{\partial}{\partial n}(\delta t)+au\frac{\partial}{\partial u}(\delta x)-au^2\frac{\partial}{\partial u}(\delta t)+av\frac{\partial}{\partial v}(\delta x)- \qquad \text{(A1-14)}$$

$$-2an\frac{\partial}{\partial n}(\delta x)-ku\frac{\partial^2}{\partial u^2}(\delta t)-ku\frac{\partial^2}{\partial v^2}(\delta t)+k\frac{\partial^2}{\partial u^2}(\delta x)+k\frac{\partial^2}{\partial v^2}(\delta x)+$$

$$+uv\frac{\partial}{\partial y}(\delta t)-u\frac{\partial}{\partial x}(\delta x)+u\frac{\partial}{\partial t}(\delta t)+u^2\frac{\partial}{\partial x}(\delta t)-v\frac{\partial}{\partial y}(\delta x)+\delta u-\frac{\partial}{\partial t}(\delta x)=0;$$

$\frac{\partial n}{\partial y}\ \frac{\partial n}{\partial u}$

$$-2kv\,\frac{\partial^2}{\partial n\partial u}(\delta t)+2k\,\frac{\partial^2}{\partial n\partial u}(\delta y)=0; \qquad \text{(A1-15)}$$

$\frac{\partial n}{\partial y}\ \frac{\partial n}{\partial u^2}$

$$-kv\,\frac{\partial^2}{\partial n^2}(\delta t)+k\,\frac{\partial^2}{\partial n^2}(\delta y)=0; \qquad \text{(A1-16)}$$

$\frac{\partial n}{\partial y}\ \frac{\partial n}{\partial v}$

$$-2kv\,\frac{\partial^2}{\partial n\partial v}(\delta t)+2k\,\frac{\partial^2}{\partial n\partial v}(\delta y)=0; \qquad \text{(A1-17)}$$

$\frac{\partial n}{\partial y}\ \frac{\partial n}{\partial v^2}$

$$-kv\,\frac{\partial^2}{\partial n^2}(\delta t)+k\,\frac{\partial^2}{\partial n^2}(\delta y)=0; \qquad \text{(A1-18)}$$

$\frac{\partial n}{\partial y}$

$$-auv\,\frac{\partial}{\partial u}(\delta t)+au\,\frac{\partial}{\partial u}(\delta y)+2avn\,\frac{\partial}{\partial n}(\delta t)+av\,\frac{\partial}{\partial v}(\delta y)-av^2\,\frac{\partial}{\partial v}(\delta t)- \qquad \text{(A1-19)}$$

$$-2an\,\frac{\partial}{\partial n}(\delta y)-kv\,\frac{\partial^2}{\partial u^2}(\delta t)-kv\,\frac{\partial^2}{\partial v^2}(\delta t)+k\,\frac{\partial^2}{\partial u^2}(\delta y)+k\,\frac{\partial^2}{\partial v^2}(\delta y)+$$

$$+uv\,\frac{\partial}{\partial x}(\delta t)-u\,\frac{\partial}{\partial x}(\delta y)-v\,\frac{\partial}{\partial y}(\delta y)v\,\frac{\partial}{\partial t}(\delta t)+v^2\,\frac{\partial}{\partial y}(\delta t)+\delta v-\frac{\partial}{\partial t}(\delta y)=0;$$

$\frac{\partial n}{\partial u}\ \frac{\partial n}{\partial v}$

$$2aku\,\frac{\partial^2}{\partial n\partial v}(\delta t)+2akv\,\frac{\partial^2}{\partial n\partial u}(\delta t)+2k\,\frac{\partial^2}{\partial n\partial u}(\delta v)+2k\,\frac{\partial^2}{\partial n\partial v}(\delta u)=0; \qquad \text{(A1-20)}$$

$\frac{\partial n}{\partial u}\ \frac{\partial n}{\partial v^2}$

$$aku\,\frac{\partial^2}{\partial n^2}(\delta t)+k\,\frac{\partial^2}{\partial n^2}(\delta u)=0; \qquad \text{(A1-21)}$$

$\frac{\partial n}{\partial u}\ \frac{\partial^2 n}{\partial u^2}$

$$2k\,\frac{\partial}{\partial n}(\delta u)+2k^2\,\frac{\partial^2}{\partial n\partial u}(\delta t)=0; \qquad \text{(A1-22)}$$

$\frac{\partial n}{\partial u}\ \frac{\partial^2 n}{\partial v^2}$

$$2k^2\,\frac{\partial^2}{\partial n\partial u}(\delta t)=0; \qquad \text{(A1-23)}$$

$\frac{\partial n}{\partial u}\ \frac{\partial^2 n}{\partial u\partial v}$

$$2k\,\frac{\partial}{\partial n}\,(\delta v)=0; \tag{A1-24}$$

$\frac{\partial n}{\partial u}\ \frac{\partial^2 n}{\partial u \partial x}$

$$2k\,\frac{\partial}{\partial n}\,(\delta x)=0; \tag{A1-25}$$

$\frac{\partial n}{\partial u}\ \frac{\partial^2 n}{\partial u \partial y}$

$$2k\,\frac{\partial}{\partial n}\,(\delta y)=0; \tag{A1-26}$$

$\frac{\partial n}{\partial u}\ \frac{\partial^2 n}{\partial t \partial u}$

$$2k\,\frac{\partial}{\partial n}\,(\delta t)=0; \tag{A1-27}$$

$\frac{\partial n}{\partial u}$

$$aku\,\frac{\partial^2}{\partial u^2}\,(\delta t)+aku\,\frac{\partial^2}{\partial v^2}\,(\delta t)+4akn\,\frac{\partial^2}{\partial n \partial u}\,(\delta t)-auv\,\frac{\partial}{\partial y}\,(\delta t)+ \tag{A1-28}$$

$$+au\,\frac{\partial}{\partial u}\,(\delta u)-au\,\frac{\partial}{\partial t}\,(\delta t)-au^2\,\frac{\partial}{\partial x}\,(\delta t)+av\,\frac{\partial}{\partial v}\,(\delta u)-2an\,\frac{\partial}{\partial n}\,(\delta u)-a\delta u+a^2uv\,\frac{\partial}{\partial v}\,(\delta t)-$$

$$-2a^2un\,\frac{\partial}{\partial n}\,(\delta t)+a^2u^2\,\frac{\partial}{\partial u}\,(\delta t)-2k\,\frac{\partial^2}{\partial n \partial u}\,(\delta n)+k\,\frac{\partial^2}{\partial u^2}\,(\delta u)+k\,\frac{\partial^2}{\partial v^2}\,(\delta u)-u\,\frac{\partial}{\partial x}\,(\delta u)-v\,\frac{\partial}{\partial y}\,(\delta u)-\frac{\partial}{\partial t}\,(\delta u)=0;$$

$\frac{\partial n}{\partial u^2}\ \frac{\partial n}{\partial v}$

$$akv\,\frac{\partial^2}{\partial n^2}\,(\delta t)+k\,\frac{\partial^2}{\partial n^2}\,(\delta v)=0; \tag{A1-29}$$

$\frac{\partial n}{\partial u^2}\ \frac{\partial^2 n}{\partial u^2}$

$$k^2\,\frac{\partial^2}{\partial n^2}\,(\delta t)=0; \tag{A1-30}$$

$\frac{\partial n}{\partial u^2}\ \frac{\partial^2 n}{\partial v^2}$

$$k^2\,\frac{\partial^2}{\partial n^2}\,(\delta t)=0; \tag{A1-31}$$

$\frac{\partial n}{\partial u^2}$

$$2aku\,\frac{\partial^2}{\partial n \partial u}\,(\delta t)+2akn\,\frac{\partial^2}{\partial n^2}\,(\delta t)-k\,\frac{\partial^2}{\partial n^2}\,(\delta n)+2k\,\frac{\partial^2}{\partial n \partial u}\,(\delta u)=0; \tag{A1-32}$$

$\frac{\partial n}{\partial u^3}$

$$aku\,\frac{\partial^2}{\partial n^2}\,(\delta t)+k\,\frac{\partial^2}{\partial n^2}\,(\delta u)=0; \tag{A1-33}$$

$\frac{\partial n}{\partial v}\ \frac{\partial^2 n}{\partial u^2}$

$$k^2\,\frac{\partial^2}{\partial n\partial v}(\delta t)+k^2\,\frac{\partial^2}{\partial n\partial v}(\delta t)=0; \qquad \text{(A1-34)}$$

$\frac{\partial n}{\partial v}\ \frac{\partial^2 n}{\partial v^2}$

$$2k\,\frac{\partial}{\partial n}(\delta v)+2k^2\,\frac{\partial^2}{\partial n\partial v}(\delta t)=0; \qquad \text{(A1-35)}$$

$\frac{\partial n}{\partial v}\ \frac{\partial^2 n}{\partial u\partial v}$

$$2k\,\frac{\partial}{\partial n}(\delta u)=0; \qquad \text{(A1-36)}$$

$\frac{\partial n}{\partial v}\ \frac{\partial^2 n}{\partial v\partial x}$

$$2k\,\frac{\partial}{\partial n}(\delta x)=0; \qquad \text{(A1-37)}$$

$\frac{\partial n}{\partial v}\ \frac{\partial^2 n}{\partial v\partial y}$

$$2k\,\frac{\partial}{\partial n}(\delta y)=0; \qquad \text{(A1-38)}$$

$\frac{\partial n}{\partial v}\ \frac{\partial^2 n}{\partial t\partial v}$

$$2k\,\frac{\partial}{\partial n}(\delta t)=0; \qquad \text{(A1-39)}$$

$\frac{\partial n}{\partial v}$

$$akv\,\frac{\partial^2}{\partial u^2}(\delta t)+akv\,\frac{\partial^2}{\partial v^2}(\delta t)+4akn\,\frac{\partial^2}{\partial n\partial v}(\delta t)- \qquad \text{(A1-40)}$$

$$-auv\,\frac{\partial}{\partial x}(\delta t)+au\,\frac{\partial}{\partial u}(\delta v)+av\,\frac{\partial}{\partial v}(\delta v)-av\,\frac{\partial}{\partial t}(\delta t)-$$

$$-av^2\,\frac{\partial}{\partial y}(\delta t)-2an\,\frac{\partial}{\partial n}(\delta v)-a\delta v+a^2uv\,\frac{\partial}{\partial u}(\delta t)-2a^2vn\,\frac{\partial}{\partial n}(\delta t)+$$

$$+a^2v^2\,\frac{\partial}{\partial v}(\delta t)+k\,\frac{\partial^2}{\partial u^2}(\delta v)-2k\,\frac{\partial^2}{\partial n\partial v}(\delta n)+k\,\frac{\partial^2}{\partial v^2}(\delta v)-u\,\frac{\partial}{\partial x}(\delta v)-v\,\frac{\partial}{\partial y}(\delta v)-\frac{\partial}{\partial t}(\delta v)=0;$$

$\frac{\partial n}{\partial v^2}\ \frac{\partial^2 n}{\partial u^2}$

$$k^2\,\frac{\partial^2}{\partial n^2}(\delta t)=0; \qquad \text{(A1-41)}$$

$\frac{\partial n}{\partial v^2}\ \frac{\partial^2 n}{\partial v^2}$

$$k^2\,\frac{\partial^2}{\partial n^2}(\delta t)=0; \qquad \text{(A1-42)}$$

$$\frac{\partial n}{\partial v^2}$$

$$2akv\,\frac{\partial^2}{\partial n \partial v}(\delta t) + 2akn\,\frac{\partial^2}{\partial n^2}(\delta t) - k\,\frac{\partial^2}{\partial n^2}(\delta n) + 2k\,\frac{\partial^2}{\partial n \partial v}(\delta v) = 0; \tag{A1-43}$$

$$\frac{\partial n}{\partial v^3}$$

$$akv\,\frac{\partial^2}{\partial n^2}(\delta t) + k\,\frac{\partial^2}{\partial n^2}(\delta v) = 0; \tag{A1-44}$$

$$\frac{\partial^2 n}{\partial u^2}$$

$$aku\,\frac{\partial}{\partial u}(\delta t) + akv\,\frac{\partial}{\partial v}(\delta t) - 2akn\,\frac{\partial}{\partial n}(\delta t) - ku\,\frac{\partial}{\partial x}(\delta t) - \tag{A1-45}$$

$$-kv\,\frac{\partial}{\partial y}(\delta t) + 2k\,\frac{\partial}{\partial u}(\delta u) - k\,\frac{\partial}{\partial t}(\delta t) + k^2\,\frac{\partial^2}{\partial u^2}(\delta t) + k^2\,\frac{\partial^2}{\partial v^2}(\delta t) = 0;$$

$$\frac{\partial^2 n}{\partial v^2}$$

$$aku\,\frac{\partial}{\partial u}(\delta t) + akv\,\frac{\partial}{\partial v}(\delta t) - 2akn\,\frac{\partial}{\partial n}(\delta t) - ku\,\frac{\partial}{\partial x}(\delta t) - \tag{A1-46}$$

$$-kv\,\frac{\partial}{\partial y}(\delta t) + 2k\,\frac{\partial}{\partial v}(\delta v) - k\,\frac{\partial}{\partial t}(\delta t) + k^2\,\frac{\partial^2}{\partial u^2}(\delta t) + k^2\,\frac{\partial^2}{\partial v^2}(\delta t) = 0;$$

$$\frac{\partial^2 n}{\partial u \partial v}$$

$$2k\,\frac{\partial}{\partial v}(\delta u) + 2k\,\frac{\partial}{\partial u}(\delta v) = 0; \tag{A1-47}$$

$$\frac{\partial^2 n}{\partial u \partial x}$$

$$2k\,\frac{\partial}{\partial u}(\delta x) = 0; \tag{A1-48}$$

$$\frac{\partial^2 n}{\partial u \partial y}$$

$$2k\,\frac{\partial}{\partial u}(\delta y) = 0; \tag{A1-49}$$

$$\frac{\partial^2 n}{\partial v \partial x}$$

$$2k\,\frac{\partial}{\partial v}(\delta x) = 0; \tag{A1-50}$$

$$\frac{\partial^2 n}{\partial v \partial y}$$

$$2k\,\frac{\partial}{\partial v}(\delta y) = 0; \tag{A1-51}$$

$$\frac{\partial^2 n}{\partial t \partial u}$$

$$2k\,\frac{\partial}{\partial u}\,(\delta t)=0; \qquad \text{(A1-52)}$$

$$\frac{\partial^2 n}{\partial t \partial v}$$

$$2k\,\frac{\partial}{\partial v}\,(\delta t)=0; \qquad \text{(A1-53)}$$

1

$$2akn\,\frac{\partial^2}{\partial u^2}\,(\delta t)+2akn\,\frac{\partial^2}{\partial v^2}\,(\delta t)-2aun\,\frac{\partial}{\partial x}\,(\delta t)-au\,\frac{\partial}{\partial u}\,(\delta n)- \qquad \text{(A1-54)}$$

$$-2avn\,\frac{\partial}{\partial y}\,(\delta t)-av\,\frac{\partial}{\partial v}\,(\delta n)+2an\,\frac{\partial}{\partial n}\,(\delta n)-2an\,\frac{\partial}{\partial t}\,(\delta t)-$$

$$-2a\delta n+2a^2un\,\frac{\partial}{\partial u}\,(\delta t)+2a^2vn\,\frac{\partial}{\partial v}\,(\delta t)-4a^2n^2\,\frac{\partial}{\partial n}\,(\delta t)-k\,\frac{\partial^2}{\partial u^2}\,(\delta n)-$$

$$-k\,\frac{\partial^2}{\partial v^2}\,(\delta n)+u\,\frac{\partial}{\partial x}\,(\delta n)+v\,\frac{\partial}{\partial y}\,(\delta n)+\frac{\partial}{\partial t}\,(\delta n)=0.$$

From (A1-37 - A1-39), (A1-48 - A1-51) we see, that $\delta x=\delta x(x,y,t)$; $\delta y=\delta y(x,y,t)$; $\delta t=\delta t(x,y,t)$. Using these expressions, we simplify the rest of equations (A1-10 - A1-54).

$$uv\,\frac{\partial}{\partial y}\,(\delta t)-u\,\frac{\partial}{\partial x}\,(\delta x)+u\,\frac{\partial}{\partial t}\,(\delta t)+u^2\,\frac{\partial}{\partial x}\,(\delta t)-v\,\frac{\partial}{\partial y}\,(\delta x)+\delta u-\frac{\partial}{\partial t}\,(\delta x)=0; \qquad \text{(A1-55)}$$

$$uv\,\frac{\partial}{\partial x}\,(\delta t)-u\,\frac{\partial}{\partial x}\,(\delta y)-v\,\frac{\partial}{\partial y}\,(\delta y)+v\,\frac{\partial}{\partial t}\,(\delta t)+v^2\,\frac{\partial}{\partial y}\,(\delta t)+\delta v-\frac{\partial}{\partial t}\,(\delta y)=0; \qquad \text{(A1-56)}$$

$$-auv\,\frac{\partial}{\partial y}\,(\delta t)+au\,\frac{\partial}{\partial u}\,(\delta u)-au\,\frac{\partial}{\partial t}\,(\delta t)-au^2\,\frac{\partial}{\partial x}\,(\delta t)+ \qquad \text{(A1-57)}$$

$$+av\,\frac{\partial}{\partial v}\,(\delta u)-a\delta u-2k\,\frac{\partial^2}{\partial n \partial u}\,(\delta n)+k\,\frac{\partial^2}{\partial u^2}\,(\delta u)+$$

$$+k\,\frac{\partial^2}{\partial v^2}\,(\delta u)-u\,\frac{\partial}{\partial x}\,(\delta u)-v\,\frac{\partial}{\partial y}\,(\delta u)-\frac{\partial}{\partial t}\,(\delta u)=0;$$

$$-k\,\frac{\partial^2}{\partial n^2}\,(\delta n)=0; \qquad \text{(A1-58)}$$

$$-auv\,\frac{\partial}{\partial x}\,(\delta t)+au\,\frac{\partial}{\partial u}\,(\delta v)+av\,\frac{\partial}{\partial v}\,(\delta v)-av\,\frac{\partial}{\partial t}\,(\delta t)- \qquad \text{(A1-59)}$$

$$-av^2\,\frac{\partial}{\partial y}\,(\delta t)-a\delta v+k\,\frac{\partial^2}{\partial u^2}\,(\delta v)-2k\,\frac{\partial^2}{\partial n \partial v}\,(\delta n)+$$

$$+k\,\frac{\partial^2}{\partial v^2}\,(\delta v)-u\,\frac{\partial}{\partial x}\,(\delta v)-v\,\frac{\partial}{\partial y}\,(\delta v)-\frac{\partial}{\partial t}\,(\delta v)=0;$$

$$-k\,\frac{\partial^2}{\partial n^2}\,(\delta n)=0; \qquad \text{(A1-60)}$$

$$-ku\frac{\partial}{\partial x}(\delta t)-kv\frac{\partial}{\partial y}(\delta t)+2k\frac{\partial}{\partial u}(\delta u)-k\frac{\partial}{\partial t}(\delta t)=0; \tag{A1-61}$$

$$-ku\frac{\partial}{\partial x}(\delta t)-kv\frac{\partial}{\partial y}(\delta t)+2k\frac{\partial}{\partial v}(\delta v)-k\frac{\partial}{\partial t}(\delta t)=0; \tag{A1-62}$$

$$2k\frac{\partial}{\partial v}(\delta u)+2k\frac{\partial}{\partial u}(\delta v)=0; \tag{A1-63}$$

$$-2aun\frac{\partial}{\partial x}(\delta t)-au\frac{\partial}{\partial u}(\delta n)-2avn\frac{\partial}{\partial y}(\delta t)-av\frac{\partial}{\partial v}(\delta n)+ \tag{A1-64}$$

$$+2an\frac{\partial}{\partial n}(\delta n)-2an\frac{\partial}{\partial t}(\delta t)-2a\delta n-k\frac{\partial^2}{\partial u^2}(\delta n)-$$

$$-k\frac{\partial^2}{\partial v^2}(\delta n)+u\frac{\partial}{\partial x}(\delta n)+v\frac{\partial}{\partial y}(\delta n)+\frac{\partial}{\partial t}(\delta n)=0.$$

From (A1-58) and (A1-60) we conclude, that

$$\delta n=A+nB; \tag{A1-65}$$

where $A=A(x,y,u,v,t), B=B(x,y,u,v,t)$. Using this expression, we simplify the rest of equations (A1-55 - A1-64).

$$uv\frac{\partial}{\partial y}(\delta t)-u\frac{\partial}{\partial x}(\delta x)+u\frac{\partial}{\partial t}(\delta t)+u^2\frac{\partial}{\partial x}(\delta t)-v\frac{\partial}{\partial y}(\delta x)+\delta u-\frac{\partial}{\partial t}(\delta x)=0; \tag{A1-66}$$

$$uv\frac{\partial}{\partial x}(\delta t)-u\frac{\partial}{\partial x}(\delta y)-v\frac{\partial}{\partial y}(\delta y)+v\frac{\partial}{\partial t}(\delta t)+v^2\frac{\partial}{\partial y}(\delta t)+\delta v-\frac{\partial}{\partial t}(\delta y)=0; \tag{A1-67}$$

$$-auv\frac{\partial}{\partial y}(\delta t)+au\frac{\partial}{\partial u}(\delta u)-au\frac{\partial}{\partial t}(\delta t)-au^2\frac{\partial}{\partial x}(\delta t)+ \tag{A1-68}$$

$$+av\frac{\partial}{\partial v}(\delta u)-a\delta u+k\frac{\partial^2}{\partial u^2}(\delta u)+k\frac{\partial^2}{\partial v^2}(\delta u)-2k\frac{\partial B}{\partial u}-u\frac{\partial}{\partial x}(\delta u)-v\frac{\partial}{\partial y}(\delta u)-\frac{\partial}{\partial t}(\delta u)=0;$$

$$-auv\frac{\partial}{\partial x}(\delta t)+au\frac{\partial}{\partial u}(\delta v)+av\frac{\partial}{\partial v}(\delta v)-av\frac{\partial}{\partial t}(\delta t)- \tag{A1-69}$$

$$-av^2\frac{\partial}{\partial y}(\delta t)-a\delta v+k\frac{\partial^2}{\partial u^2}(\delta v)+k\frac{\partial^2}{\partial v^2}(\delta v)-2k\frac{\partial B}{\partial v}-u\frac{\partial}{\partial x}(\delta v)-v\frac{\partial}{\partial y}(\delta v)-\frac{\partial}{\partial t}(\delta v)=0;$$

$$-ku\frac{\partial}{\partial x}(\delta t)-kv\frac{\partial}{\partial y}(\delta t)+2k\frac{\partial}{\partial u}(\delta u)-k\frac{\partial}{\partial t}(\delta t)=0; \tag{A1-70}$$

$$-ku\frac{\partial}{\partial x}(\delta t)-kv\frac{\partial}{\partial y}(\delta t)+2k\frac{\partial}{\partial v}(\delta v)-k\frac{\partial}{\partial t}(\delta t)=0; \tag{A1-71}$$

$$2k\frac{\partial}{\partial v}(\delta u)+2k\frac{\partial}{\partial u}(\delta v)=0; \tag{A1-72}$$

$$-2au\frac{\partial}{\partial x}(\delta t)-au\frac{\partial B}{\partial u}-2av\frac{\partial}{\partial y}(\delta t)-av\frac{\partial B}{\partial v}-2a\frac{\partial}{\partial t}(\delta t)- \tag{A1-73}$$

$$-k\frac{\partial^2 B}{\partial u^2}-k\frac{\partial^2 B}{\partial v^2}+u\frac{\partial B}{\partial x}+v\frac{\partial B}{\partial y}+\frac{\partial B}{\partial t}=0;$$

$$-au\frac{\partial A}{\partial u}-av\frac{\partial A}{\partial v}-2aA-k\frac{\partial^2 A}{\partial u^2}--k\frac{\partial^2 A}{\partial v^2}+u\frac{\partial A}{\partial x}+v\frac{\partial A}{\partial y}+\frac{\partial A}{\partial t}=0. \tag{A1-74}$$

Equation (A1-74) is simply Fokker - Planck equation for A.

We solve (A1-66 - A1-67) and find $\delta u, \delta v$

$$\delta u=-(uv\frac{\partial}{\partial y}(\delta t)-u\frac{\partial}{\partial x}(\delta x)+u\frac{\partial}{\partial t}(\delta t)+u^2\frac{\partial}{\partial x}(\delta t)-v\frac{\partial}{\partial y}(\delta x)-\frac{\partial}{\partial t}(\delta x)); \tag{A1-76}$$

$$\delta v=-(uv\frac{\partial}{\partial x}(\delta t)-u\frac{\partial}{\partial x}(\delta y)-v\frac{\partial}{\partial y}(\delta y)+v\frac{\partial}{\partial t}(\delta t)+v^2\frac{\partial}{\partial y}(\delta t)-\frac{\partial}{\partial t}(\delta y)). \tag{A1-77}$$

This gives for (A1-67 - A1-73)

$$-2auv\frac{\partial}{\partial y}(\delta t)-au\frac{\partial}{\partial t}(\delta t)-2au^2\frac{\partial}{\partial x}(\delta t)-a\frac{\partial}{\partial t}(\delta x)-2k\frac{\partial}{\partial x}(\delta t)- \tag{A1-78}$$

$$-2k\frac{\partial B}{\partial u}+2uv\frac{\partial^2}{\partial t\partial y}(\delta t)-2uv\frac{\partial^2}{\partial x\partial y}(\delta x)+uv^2\frac{\partial^2}{\partial y^2}(\delta t)+u\frac{\partial^2}{\partial t^2}(\delta t)-2u\frac{\partial^2}{\partial t\partial x}(\delta x)+$$

$$+2u^2v\frac{\partial^2}{\partial x\partial y}(\delta t)+2u^2\frac{\partial^2}{\partial t\partial x}(\delta t)-u^2\frac{\partial^2}{\partial x^2}(\delta x)+u^3\frac{\partial^2}{\partial x^2}(\delta t)-2v\frac{\partial^2}{\partial t\partial y}(\delta x)-v^2\frac{\partial^2}{\partial y^2}(\delta x)-\frac{\partial^2}{\partial t^2}(\delta x)=0;$$

$$-2auv\frac{\partial}{\partial x}(\delta t)-av\frac{\partial}{\partial t}(\delta t)-2av^2\frac{\partial}{\partial y}(\delta t)-a\frac{\partial}{\partial t}(\delta y)-2k\frac{\partial}{\partial y}(\delta t)- \tag{A1-79}$$

$$-2k\frac{\partial B}{\partial v}+2uv\frac{\partial^2}{\partial t\partial x}(\delta t)-2uv\frac{\partial^2}{\partial x\partial y}(\delta y)+2uv^2\frac{\partial^2}{\partial x\partial y}(\delta t)-2u\frac{\partial^2}{\partial t\partial x}(\delta y)+$$

$$+u^2v\frac{\partial^2}{\partial x^2}(\delta t)-u^2\frac{\partial^2}{\partial x^2}(\delta y)+v\frac{\partial^2}{\partial t^2}(\delta t)-2v\frac{\partial^2}{\partial t\partial y}(\delta y)+2v^2\frac{\partial^2}{\partial t\partial y}(\delta t)-v^2\frac{\partial^2}{\partial y^2}(\delta y)+v^3\frac{\partial^2}{\partial y^2}(\delta t)-\frac{\partial^2}{\partial t^2}(\delta y)=0;$$

$$-5ku\frac{\partial}{\partial x}(\delta t)-3kv\frac{\partial}{\partial y}(\delta t)+2k\frac{\partial}{\partial x}(\delta x)-3k\frac{\partial}{\partial t}(\delta t)=0; \tag{A1-80}$$

$$-3ku\frac{\partial}{\partial x}(\delta t)-5kv\frac{\partial}{\partial y}(\delta t)+2k\frac{\partial}{\partial y}(\delta y)-3k\frac{\partial}{\partial t}(\delta t)=0; \tag{A1-81}$$

$$-2ku\frac{\partial}{\partial y}(\delta t)-2kv\frac{\partial}{\partial x}(\delta t)+2k\frac{\partial}{\partial y}(\delta x)+2k\frac{\partial}{\partial x}(\delta y)=0; \tag{A1-82}$$

$$-2au\frac{\partial}{\partial x}(\delta t)-au\frac{\partial B}{\partial u}-2av\frac{\partial}{\partial y}(\delta t)-av\frac{\partial B}{\partial v}-2a\frac{\partial}{\partial t}(\delta t)- \tag{A1-83}$$

$$-k\frac{\partial^2 B}{\partial u^2}-k\frac{\partial^2 B}{\partial v^2}+u\frac{\partial B}{\partial x}+v\frac{\partial B}{\partial y}+\frac{\partial B}{\partial t}=0.$$

Now we can collect similar terms in (A1-80 - A1-82) and so split them into nine equations:

$$-5k\frac{\partial}{\partial x}(\delta t)=0; \tag{A1-84}$$

$$-3k\frac{\partial}{\partial y}(\delta t)=0; \tag{A1-85}$$

$$2k\,\frac{\partial}{\partial x}\,(\delta x) - 3k\,\frac{\partial}{\partial t}\,(\delta t) = 0; \tag{A1-86}$$

$$-3k\,\frac{\partial}{\partial x}\,(\delta t) = 0; \tag{A1-87}$$

$$-5k\,\frac{\partial}{\partial y}\,(\delta t) = 0; \tag{A1-88}$$

$$2k\,\frac{\partial}{\partial y}\,(\delta y) - 3k\,\frac{\partial}{\partial t}\,(\delta t) = 0; \tag{A1-89}$$

$$-2k\,\frac{\partial}{\partial y}\,(\delta t) = 0; \tag{A1-90}$$

$$-2k\,\frac{\partial}{\partial x}\,(\delta t) = 0; \tag{A1-91}$$

$$2k\,\frac{\partial}{\partial y}\,(\delta x) + 2k\,\frac{\partial}{\partial x}\,(\delta y) = 0. \tag{A1-92}$$

From (A1-84 - A1-85), (A1-87 - A1-88), (A1-90 - A1-91) we see, that $\delta t = \delta t(t)$, which results in further simplifications

$$-au\,\frac{\partial}{\partial t}\,(\delta t) - a\,\frac{\partial}{\partial t}\,(\delta x) - 2k\,\frac{\partial B}{\partial u} - 2uv\,\frac{\partial^2}{\partial x\partial y}\,(\delta x) + u\,\frac{\partial^2}{\partial t^2}\,(\delta t) - 2u\,\frac{\partial^2}{\partial t\partial x}\,(\delta x) - \tag{A1-93}$$

$$-u^2\,\frac{\partial^2}{\partial x^2}\,(\delta x) - 2v\,\frac{\partial^2}{\partial t\partial y}\,(\delta x) - v^2\,\frac{\partial^2}{\partial y^2}\,(\delta x) - \frac{\partial^2}{\partial t^2}\,(\delta x) = 0;$$

$$-av\,\frac{\partial}{\partial t}\,(\delta t) - a\,\frac{\partial}{\partial t}\,(\delta y) - 2k\,\frac{\partial B}{\partial v} - 2uv\,\frac{\partial^2}{\partial x\partial y}\,(\delta y) - 2u\,\frac{\partial^2}{\partial t\partial x}\,(\delta y) - \tag{A1-94}$$

$$-u^2\,\frac{\partial^2}{\partial x^2}\,(\delta y) + v\,\frac{\partial^2}{\partial t^2}\,(\delta t) - 2v\,\frac{\partial^2}{\partial t\partial y}\,(\delta y) - v^2\,\frac{\partial^2}{\partial y^2}\,(\delta y) - \frac{\partial^2}{\partial t^2}\,(\delta y) = 0;$$

$$2k\,\frac{\partial}{\partial x}\,(\delta x) - 3k\,\frac{\partial}{\partial t}\,(\delta t) = 0; \tag{A1-95}$$

$$2k\,\frac{\partial}{\partial y}\,(\delta y) - 3k\,\frac{\partial}{\partial t}\,(\delta t) = 0; \tag{A1-96}$$

$$2k\,\frac{\partial}{\partial y}\,(\delta x) + 2k\,\frac{\partial}{\partial x}\,(\delta y) = 0; \tag{A1-97}$$

$$-au\,\frac{\partial B}{\partial u} - av\,\frac{\partial B}{\partial v} - 2a\,\frac{\partial}{\partial t}\,(\delta t) - k\,\frac{\partial^2 B}{\partial u^2} - k\,\frac{\partial^2 B}{\partial v^2} + u\,\frac{\partial B}{\partial x} + v\,\frac{\partial B}{\partial y} + \frac{\partial B}{\partial t} = 0. \tag{A1-98}$$

We integrate (A1-95 - A1-96) and find

$$\delta x = C + 3/2x\,\frac{\partial}{\partial t}\,(\delta t); \tag{A1-99}$$

$$\delta y = D + 3/2y\,\frac{\partial}{\partial t}\,(\delta t); \tag{A1-100}$$

where $C = C(y,t)$, $D = D(x,t)$. We substitute these expressions to (A1-93 - A1-94), (A1-97 - A1-98) and obtain

$$-3/2ax \frac{\partial^2}{\partial t^2}(\delta t) - au \frac{\partial}{\partial t}(\delta t) - a \frac{\partial C}{\partial t} - 2k \frac{\partial B}{\partial u} - 3/2x \frac{\partial^3}{\partial t^3}(\delta t) - \tag{A1-101}$$

$$-2u \frac{\partial^2}{\partial t^2}(\delta t) - 2v \frac{\partial^2 C}{\partial t \partial y} - v^2 \frac{\partial^2 C}{\partial y^2} - \frac{\partial^2 C}{\partial t^2} = 0;$$

$$-3/2ay \frac{\partial^2}{\partial t^2}(\delta t) - av \frac{\partial}{\partial t}(\delta t) - a \frac{\partial D}{\partial t} - 2k \frac{\partial B}{\partial v} - 3/2y \frac{\partial^3}{\partial t^3}(\delta t) - \tag{A1-102}$$

$$-2u \frac{\partial^2 D}{\partial t \partial x} - u^2 \frac{\partial^2 D}{\partial x^2} - 2v \frac{\partial^2}{\partial t^2}(\delta t) - \frac{\partial^2 D}{\partial t^2} = 0;$$

$$2k \frac{\partial C}{\partial y} + 2k \frac{\partial D}{\partial x} = 0; \tag{A1-103}$$

$$-au \frac{\partial B}{\partial u} - av \frac{\partial B}{\partial v} - 2a \frac{\partial}{\partial t}(\delta t) - k \frac{\partial^2 B}{\partial u^2} - k \frac{\partial^2 B}{\partial v^2} + u \frac{\partial B}{\partial x} + v \frac{\partial B}{\partial y} + \frac{\partial B}{\partial t} = 0. \tag{A1-104}$$

We find $\frac{\partial B}{\partial u}$ from (A1-101) and $\frac{\partial B}{\partial v}$ from (A1-102):

$$\frac{\partial B}{\partial u} = \frac{1}{k}(-3/4ax \frac{\partial^2}{\partial t^2}(\delta t) - 1/2au \frac{\partial}{\partial t}(\delta t) - 1/2a \frac{\partial C}{\partial t} - 3/4x \frac{\partial^3}{\partial t^3}(\delta t) - \tag{A1-105}$$

$$-u \frac{\partial^2}{\partial t^2}(\delta t) - v \frac{\partial^2 C}{\partial t \partial y} - 1/2v^2 \frac{\partial^2 C}{\partial y^2} - 1/2 \frac{\partial^2 C}{\partial t^2});$$

$$\frac{\partial B}{\partial v} = \frac{1}{k}(-3/4ay \frac{\partial^2}{\partial t^2}(\delta t) - 1/2av \frac{\partial}{\partial t}(\delta t) - 1/2a \frac{\partial D}{\partial t} - 3/4y \frac{\partial^3}{\partial t^3}(\delta t) - \tag{A1-106}$$

$$-u \frac{\partial^2 D}{\partial t \partial x} - 1/2u^2 \frac{\partial^2 D}{\partial x^2} - v \frac{\partial^2}{\partial t^2}(\delta t) - 1/2 \frac{\partial^2 D}{\partial t^2}).$$

Differentiating (A1-105) by v we have

$$\frac{\partial^2 B}{\partial u \partial v} = \frac{1}{k}(-v \frac{\partial^2 C}{\partial y^2} - \frac{\partial^2 C}{\partial t \partial y}); \tag{A1-107}$$

Differentiating (A1-106) by u we have

$$\frac{\partial^2 B}{\partial u \partial v} = \frac{1}{k}(-u \frac{\partial^2 D}{\partial x^2} - \frac{\partial^2 D}{\partial t \partial x}). \tag{A1-108}$$

We know, that $C = C(y,t)$, $D = D(x,t)$ and so we conclude from (A1-107 - A1-108)

$$C = C_1 y + E; \tag{A1-109}$$

$$D = C_2 x + F; \tag{A1-110}$$

where $E = E(t), F = F(t), C_1 + C_2 = 0$.

We find derivatives of B.

$$\frac{\partial B}{\partial u} = \frac{1}{2k}(3/2ax \frac{\partial^2}{\partial t^2}(\delta t) + au \frac{\partial}{\partial t}(\delta t) + a \frac{\partial E}{\partial t} - 3/2x \frac{\partial^3}{\partial t^3}(\delta t) - 2u \frac{\partial^2}{\partial t^2}(\delta t) - \frac{\partial^2 E}{\partial t^2}); \tag{A1-111}$$

$$\frac{\partial B}{\partial v}=\frac{1}{2k}\,(3/2ay\,\frac{\partial^2}{\partial t^2}\,(\delta t)+av\,\frac{\partial}{\partial t}\,(\delta t)+a\,\frac{\partial F}{\partial t}-3/2y\,\frac{\partial^3}{\partial t^3}\,(\delta t)-2v\,\frac{\partial^2}{\partial t^2}\,(\delta t)-\frac{\partial^2 F}{\partial t^2}); \quad \text{(A1-112)}$$

$$\frac{\partial^2 B}{\partial u\partial v}=\frac{\partial^2 B}{\partial v\partial u}=0. \quad \text{(A1-113)}$$

Integration of (A1-111 - A1-112) gives

$$B=G+\frac{1}{2k}\,(3/2uax\,\frac{\partial^2}{\partial t^2}\,(\delta t)+1/2au^2\,\frac{\partial}{\partial t}\,(\delta t)+ua\,\frac{\partial E}{\partial t}-3/2ux\,\frac{\partial^3}{\partial t^3}\,(\delta t)-u^2\,\frac{\partial^2}{\partial t^2}\,(\delta t)-u\,\frac{\partial^2 E}{\partial t^2})+ \quad \text{(A1-114)}$$

$$+\frac{1}{2k}\,(3/2vay\,\frac{\partial^2}{\partial t^2}\,(\delta t)+1/2av^2\,\frac{\partial}{\partial t}\,(\delta t)+va\,\frac{\partial F}{\partial t}-3/2vy\,\frac{\partial^3}{\partial t^3}\,(\delta t)-v^2\,\frac{\partial^2}{\partial t^2}\,(\delta t)-v\,\frac{\partial^2 F}{\partial t^2});$$

where $G=G(x,y,t)$.

Substitution of (A1-114) to (A1-98), collecting and equating to zero similar terms by u,v gives

$$3/4a^2k^{-1}x\,\frac{\partial^2}{\partial t^2}\,(\delta t)+1/2a^2k^{-1}\,\frac{\partial E}{\partial t}-3/4k^{-1}x\,\frac{\partial^4}{\partial t^4}\,(\delta t)-1/2k^{-1}\,\frac{\partial^3 E}{\partial t^3}+\frac{\partial G}{\partial x}=0; \quad \text{(A1-115)}$$

$$1/2a^2k^{-1}\,\frac{\partial}{\partial t}\,(\delta t)-5/4k^{-1}\,\frac{\partial^3}{\partial t^3}\,(\delta t)=0; \quad \text{(A1-116)}$$

$$3/4a^2k^{-1}y\,\frac{\partial^2}{\partial t^2}\,(\delta t)+1/2a^2k^{-1}\,\frac{\partial F}{\partial t}-3/4k^{-1}y\,\frac{\partial^4}{\partial t^4}\,(\delta t)-1/2k^{-1}\,\frac{\partial^3 F}{\partial t^3}+\frac{\partial G}{\partial y}=0; \quad \text{(A1-117)}$$

$$1/2a^2k^{-1}\,\frac{\partial}{\partial t}\,(\delta t)-5/4k^{-1}\,\frac{\partial^3}{\partial t^3}\,(\delta t)=0; \quad \text{(A1-118)}$$

$$-a\,\frac{\partial}{\partial t}\,(\delta t)+2\,\frac{\partial^2}{\partial t^2}\,(\delta t)+\frac{\partial G}{\partial t}=0; \quad \text{(A1-119)}$$

Integrate (A1-115), (A1-117) and obtain following expression ($H=H(t)$):

$$G=H-\frac{1}{k}\left(3/8a^2x^2\,\frac{\partial^2}{\partial t^2}\,(\delta t)+1/2xa^2\,\frac{\partial E}{\partial t}-3/8x^2\,\frac{\partial^4}{\partial t^4}\,(\delta t)-1/2x\,\frac{\partial^3 E}{\partial t^3}\right)- \quad \text{(A1-120)}$$

$$-\frac{1}{k}\left(3/8a^2y^2\,\frac{\partial^2}{\partial t^2}\,(\delta t)+1/2ya^2\,\frac{\partial F}{\partial t}-3/8y^2\,\frac{\partial^4}{\partial t^4}\,(\delta t)-1/2y\,\frac{\partial^3 F}{\partial t^3}\right).$$

Substitution of (A1-120) to (A1-119), collecting and equating to zero terms by x,y gives

$$-1/2a^2k^{-1}\,\frac{\partial^2 E}{\partial t^2}+1/2k^{-1}\,\frac{\partial^4 E}{\partial t^4}=0; \quad \text{(A1-121)}$$

$$-3/8a^2k^{-1}\,\frac{\partial^3}{\partial t^3}\,(\delta t)+3/8k^{-1}\,\frac{\partial^5}{\partial t^5}\,(\delta t)=0; \quad \text{(A1-122)}$$

$$-1/2a^2k^{-1}\,\frac{\partial^2 F}{\partial t^2}+1/2k^{-1}\,\frac{\partial^4 F}{\partial t^4}=0; \quad \text{(A1-123)}$$

$$-3/8a^2k^{-1}\,\frac{\partial^3}{\partial t^3}\,(\delta t)+3/8k^{-1}\,\frac{\partial^5}{\partial t^5}\,(\delta t)=0; \quad \text{(A1-124)}$$

$$-a\,\frac{\partial}{\partial t}\,(\delta t)+2\,\frac{\partial^2}{\partial t^2}\,(\delta t)+\frac{\partial H}{\partial t}=0. \quad \text{(A1-125)}$$

From (A1-116), (A1-118), (A1-122), (A1-124) we conclude, that

$$\delta t = const = C_3. \tag{A1-126}$$

From (A1-121), (A1-123) we conclude, that

$$E = C_4 + C_5 t + C_6 e^{-at} + C_7 e^{at}; \tag{A1-127}$$

$$F = C_8 + C_9 t + C_{10} e^{-at} + C_{11} e^{at}; \tag{A1-128}$$

From (A1-125) and (A1-126) we see, that

$$H = C_{12}. \tag{A1-129}$$

We obtain using (A1-126 - A1-129) and backward substitution final expressions for variations:

$$\delta n = -1/2ak^{-1}unC_5 - 1/2ak^{-1}vnC_9 - 1/2a^2k^{-1}xnC_5 - \tag{A1-130}$$

$$-1/2a^2k^{-1}ynC_9 - a^2k^{-1}unC_7e^{at} - a^2k^{-1}vnC_{11}e^{at} + nC_{12} + A;$$

$$\delta x = yC_1 + tC_5 + C_4 + C_6 e^{-at} + C_7 e^{at}; \tag{A1-131}$$

$$\delta y = xC_2 + tC_9 + C_8 + C_{10} e^{-at} + C_{11} e^{at}; \tag{A1-132}$$

$$\delta u = -aC_6 e^{-at} + aC_7 e^{at} + vC_1 + C_5; \tag{A1-133}$$

$$\delta v = -aC_{10} e^{-at} + aC_{11} e^{at} + uC_2 + C_9; \tag{A1-134}$$

$$\delta t = C_3. \tag{A1-135}$$

This ends calculations.